Kyiv School of Economics

# Commutative Algebra

**2025/2026, Autumn Trimester**

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| **Course faculty** | Piotr Achinger<br>e-mail: pachinger@kse.org.ua |
| **Department** | Department of Mathematics |
| **Study program** | Masters in Mathematics |
| **ECTS credits** | 6 (180 hours) |
| **Class hours** | 60 hours, 20 lectures and 10 practices (80 min each)<br>The course starts on October 14 |
| **Course language** | English |
| **Course format** | Offline, with remote participation capabilities |

# Overview

## Prerequisites

This is an advanced (Master level) class in pure mathematics, with a fair amount of theory building. Prior experience with proof-based courses in math is necessary. That said, the class does not have heavy prerequisites and should be accessible to undergraduates interested in the topic.

Both undergraduates and PhD students are welcome to attend if they meet the prerequisites. If you are interested in taking the course but aren't sure if this is the right time, consult with your mentor or the lecturer.

It is a good idea to take this class together with *MATH530 Sheaves and Cohomology* (4 weeks, Sep 8–Oct 10), especially if you want to take *MATH551 Algebraic Geometry* in Spring.

I will assume the students know linear algebra and topology and have basic familiarity with groups, rings, and fields. Some material from *MATH530 Sheaves and Cohomology* (sheaves, basic homological algebra) may be helpful but is not required.

**Dependent courses.** The course is a highly recommended prerequisite for *MATH551 Algebraic Geometry*, *MATH552 Modular Forms and Elliptic Curves*, and *MATH553 Algebraic Number Theory*.

## Background and course rationale

Commutative algebra is the study of commutative rings and modules over them. It serves as a foundation of algebraic geometry and algebraic number theory. Typical examples of commutative rings are the ring of integers **Z** and the ring of polynomials $k[x_1, \dots, x_n]$ in $n$ variables over a field $k$ (we will encounter lots of more interesting examples). Geometrically, the latter can be thought of as the ring of polynomial functions on the $n$-dimensional space. Such a geometric perspective is very useful, enabling us to use our visual intuition when working with the polynomial ring. It turns out that one can view any commutative ring geometrically: to a commutative ring $R$ we can associate a topological space $\mathrm{Spec}(R)$, called its spectrum, whose geometry reflects the algebraic properties of $R$. Thus (unlike elsewhere in algebra, e.g. for noncommutative rings), in commutative algebra one can mix and match the precision and formality of algebra with geometric intuition.

### Course aims

The goal of the course is to introduce the basic concepts of commutative algebra, such as modules, local rings, valuation rings, Noetherian rings, completion, the Krull dimension etc., emphasizing the geometric meaning behind them when possible. A more detailed list of topics can be found below.

### Learning outcomes

Having passed this course, students will command the necessary skills and knowledge to easily learn algebraic geometry, including the theory of schemes. They will be able to compute easily and efficiently with commutative rings and modules which commonly appear in practice.

## Course Structure

Every week there will be a problem sheet (homework). It is very important to attempt homework assignments regularly, in order to proceed with the understanding of the material in this course and obtain feedback about one's understanding of the subject. During **practical sessions** students report their solutions to homework assignments and brainstorm on similar questions. To obtain the full mark for their work in practical sessions, students should present correct answers along with their proofs.

At the end of the course there will be a **final exam** lasting for 3 hours.

***Course duration.*** 10 weeks (Oct 13–Dec 19), 6 ECTS (180 hours)

***Lectures.*** Two 80 min lectures per week.

***Practices.*** One 80 min practice session per week devoted mostly to discussing the homework assignments (see below).

***Homework assignments.*** Weekly homework assignments (9 problem lists in total), published right after the practice session and due on the next practice session. Each student picks three problems from the list and reports which problems they have solved during the practice session. They may then be asked to present their solutions in class. Each problem is worth 1 point, adding up to 27 points. Extra exercises (marked with a *) are to be submitted in writing for extra credit. The total number of points for homework assignments is capped at 30.

***Final exam.*** The final written exam will take place in class. It will last 3 hours and consist of about 5 problems, worth 30 points in total.

The **final grade** will depend on the total score (out of 60 points).

# Course Faculty

**Piotr Achinger**, Simons Professor in Mathematics at KSE
Academic Director of Graduate Programs in Mathematics
Associate Professor at IMPAN

**Office hours:** Tuesday, 10:00-12:00, KSE Dragon Capital Building, 3 Shpaka St., Front staircase, floor 5½

Piotr works in algebraic geometry and is broadly interested in the topological properties of algebraically defined geometric objects. He works at the Institute of Mathematics of the Polish Academy of Sciences (IMPAN) in Warsaw (on leave during the 2025/26 academic year). Він розмовляє українською.

Piotr earned his PhD from the University of California, Berkeley, in 2015.

After holding postdoctoral positions at the Banach Center in Warsaw and the Institut des Hautes Études Scientifiques (IHES) in Paris, he became a researcher at the Institute of Mathematics of the Polish Academy of Sciences (IMPAN).

Piotr obtained his habilitation in 2022.

https://achinger.impan.pl/

# Course Plan

### Reading list

There is no required textbook. The course will be partially based on lecture notes by Joachim Jelisiejew (University of Warsaw), which will be shared with the class when the course starts. Other recommended sources are:

1. M. F. Atiyah, I. G. Macdonald *Introduction to Commutative Algebra*, Addison–Wesley 1969
2. D. Eisenbud *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer 2004

**Course plan (tentative)**

1. Preliminaries about commutative rings (principal ideal domains, quotient rings). Basic category theory. Review of Galois theory (if necessary).
2. The spectrum of a commutative ring. Localizations and spectra. Nilradicals.
3. Modules, localization of modules. Nakayama's lemma. Tensor products of rings and modules. Flatness. Modules of differentials.
4. Chain conditions. Noetherian and Artinian rings.
5. Graded rings and modules.
6. Noether normalization, Nullstellensatz, and their corollaries.
7. Finite and integral ring homomorphisms. Normal rings. Finiteness of integral closure. Valuation rings and valuations.
8. Krull dimension. Dimension and transcendence degree.
9. Discrete valuation rings and Dedekind domains. Modules over PIDs. Ideal class group.
10. Completion. Complete local rings.